Turing Machines and Undecidability with Special Focus on Computer Viruses

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Abstract

In this paper certain aspects of computability theory will be discussed in relation to computer viruses. For instance the undecidability of detection of computer viruses will be examined.

Contents

1 Introduction

The purpose of the present paper is twofold. First, to introduce the essential terminology for discussing unsolvable problems, and second, by using this terminology, to give a formal description of computer viruses.

In our everyday lifes we are confronted with problems and usually we are able to solve them. Solvable problems are so frequent that we might not even reflect about the fact that there are unsolvable problems, problems that in no way are possible to solve at all. This is indeed an intriguing thought and in Section 2 this issue will be discussed in great detail. Section 3 involves a discussion of the development of computers and this section is a bridge between the theory of computability in Section 2 and the theory of computer viruses in Section 4. Finally, in Section 5 a summary is given with an informal description of computer viruses.

A large part of the present paper is based on the thesis by Fred Cohen [1]. The material from the thesis that will be covered here (computational aspects of computer viruses) has also been published elsewhere [2]. Fred Cohen is most well known for his groundbreaking work in computer viruses, where he did the first indepth mathematical analysis. In this paper a slightly different terminology than the one used by Fred Cohen will be employed, but the contents should be essentially the same. Fred Cohen has also performed many startling experiments with computer viruses and he developed the first protection mechanisms, many of which got a widespread use. Protection mechanisms will not be discussed in this paper, however they will be mentioned in the summary.

2 Computability theory

Computability theory (also known as recursion theory), originated with the seminal work of Gödel, Church, Turing, Kleene, and Post in the 1930's, and includes a wide spectrum of topics, of which a few will be covered here.

2.1 Theory of algorithms

Informally speaking an algorithm is a collection of simple instructions for carrying out some task in a finite number of steps. Algorithms play a vital role in our everyday lifes where they usually have other names like procedures or recipes. However, it is in mathematics that the notion of algorithm has become one of the central concepts. As a matter of fact the word algorithm derives from the name of the famous Arab mathematician al-Khowarizmi who wrote a book about methods for arithmetical calculation in the 800s. Moreover, the term algorithm has become a general scientific and technological concept used in a variety of areas. A popular point of view on algorithm is presented by Rogers [3]:

> Algorithm is a clerical (i.e., deterministic, book-keeping) procedure which can be applied to any of a certain class of symbolic inputs and which will eventually yield, for each such input, a corresponding output.

Even though algorithms have had a long history in mathematics, the notion of algorithm itself was not defined precisely until the twentieth century. Before that one had to rely upon an intuitive notion like the one above. But an informal notion is insufficient for solving certain problems, for instance Hilbert's tenth problem[1]. The intuitive concept of algorithm may be adequate for giving algorithms for certain tasks, but it is useless for showing that no algorithm exists for a particular task. Therefore an exact mathematical concept of algorithm is necessary. This is provided by the theory of algorithms which (according to Burgin [5]) constitutes one of the major achievements of the twentieth century mathematics. The main accomplishment of this theory has been the elaboration of an exact mathematical model of algorithm. Different mathematicians have suggested different models: Turing machines, λ-calculus, partial recursive functions, Post productions, Kolmogorov algorithms, finite automata, vector machines, register machines, neural networks, Minsky machines, etc. [5]. It is not the purpose of this paper to discuss the various models of algorithm and the reason for listing them in the previous sentence is just to demonstrate the plentitude of models existing.

The most popular model of algorithm was suggested by the English mathematician Alan Turing. Consequently, it is called a Turing machine. This abstract device will be discussed in great detail in the next subsection. The definition of a Turing machine came in 1936 and in the very same year Alonzo Church presented work in the same area using a notational system called the λ-calculus to define algorithms. These two definitions were shown to be equivalent. Other definitions of algorithm, like the finite automata, have shown to be weaker than the Turing machine, i.e., Turing machines can compute everything finite automatas can compute, but there are things computable by Turing machines which cannot be computed by finite automatas.

[1]Hilbert's tenth problem was to devise an algorithm that tests whether a polynomial has an integral root. Hilbert did not use the term algorithm but rather the informal: "a process according to which it can be determined by a finite number of operations" [4].

In spite of all differences, it has been proven that each mathematical model of algorithm is either equivalent or weaker to Turing machines. The connection between the informal notion of algorithm and the precise definition has come to be called the Church-Turing thesis (see Figure 1).

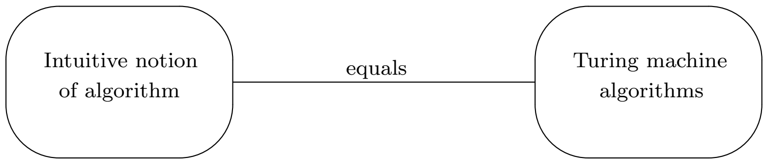


Figure 1: *The Church-Turing Thesis*

The Church-Turing thesis provides the definition of algorithm necessary to resolve Hilbert's tenth problem, and in 1970, Yuri Matijasevič showed that no algorithm exists for testing whether a polynomial has integral roots.

2.2 The Turing machine

This is what Turing supplied in working out a definition of algorithm. He analysed what could be achieved by a person performing a methodical process, and seizing on the idea of something done "mechanically", expressed the analysis in terms of a theoretical machine able to perform certain precisely defined elementary operations on symbols on paper tape [6]. By continuing citing Hodges [6], Turing's "work emerged as that of a complete outsider" and his "originality lay in seeing the relevance of mathematical logic to a problem originally seen as one of physics.[...] Turing made a bridge between the logical and the physical worlds, thought and action, which crossed conventional boundaries."

2.2.1 Definition

In the definition the setup in Hopcroft *et al.* [7] will be followed. However, a semi-infinite tape will be used in order to be as close as possible to the definition used in the thesis of Fred Cohen [1]. In the previous statement it is indicated that a Turing machine can be defined in several ways. The one that is going to be employed in this paper is for a deterministic Turing machine with one semi-infinite tape. In Hopcroft [7] and Sipser [4] several other definitions, like multitape Turing machines, nondeterministic Turing machines, multistack machines, counter machines, etc., are discussed.

A Turing machine can be visualized as in Figure 2. The machine consists of a *finite control*, which can be in any of a finite number of states (q_i). There is a semi-infinite *tape* divided into squares or cells. Each cell can hold any of a finite number of symbols. In Figure 2 the cells are numbered from 0 and upwards. Finally, there is a *tape head* that is always positioned at one of the tape cells. The Turing machine is said to be scanning that cell.

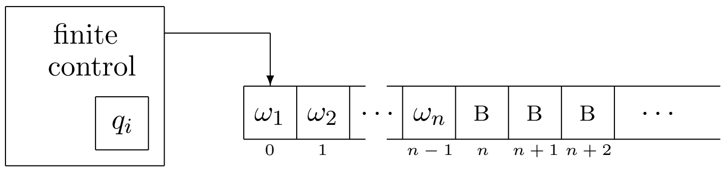


Figure 2: *A Turing machine*

A move of a Turing machine is a function of the finite state control and the tape symbol scanned. In one move the Turing machine will:

1. Change state (optionally).
2. Write a tape symbol in the cell scanned (optionally).
3. Move the tape head left or right. (If the tape head is at the lefthand end of the tape it can not move left but remains at the same position although the instruction is to move left.)

Definition 2.1 *In a formal notation a Turing machine is a 7-tuple,* $(Q, \Sigma, \Gamma, \delta, q_0, \text{B}, F)$*, where* Q*,* Σ*, and* Γ *are all finite sets and:*

1. Q *is the set of states.*
2. Σ *is the input alphabet not containing the special blank symbol* B*.*
3. Γ *is the tape alphabet, where* $\text{B} \in \Gamma$ *and* $\Sigma \subset \Gamma$*.*
4. δ*:* $Q \times \Gamma \rightarrow Q \times \Gamma \times$ $\{L,R\}$ *is the transition function defining the moves (L and R denote left and right, respectively).*
5. $q_0 \in Q$ *is the start state.*
6. B *is the blank symbol.*
7. $F \in Q$ *is the set of final or accepting states.*

A Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, \text{B}, F)$ computes as follows. Initially M receives its input $\omega = \omega_1\omega_2 \ldots \omega_n \in \Sigma^*$ on the leftmost n cells of the tape, and the rest of the tape filled with blank symbols (see Figure 2). Since the blank symbol is not included in the input alphabet the first blank appearing on the tape marks the end of the input. The tape head starts on the leftmost cell of the tape and the finite control is in the start state. A setting of these three items (the tape, the tape head, and the state of the finite control) is called a *configuration* of the Turing machine. With $q_i = q_0$ the Turing machine in Figure 2 is in its initial configuration with input ω. Once M starts, the computation proceeds according to the rules described by the transition function. If M ever tries to move its head to the left off the left-hand end of the tape, the head stays in the same place for that move, even though the transition function indicates L.

As a Turing machine computes its configuration is changing. Configurations are often represented in a special way. For a state q and two strings u and v over the tape alphabet Γ uqv expresses the configuration where the current state is q, the current tape contents is uv, and the current head location is the first symbol of v (see Figure 3). The tape contains only blanks following the last symbol of v.

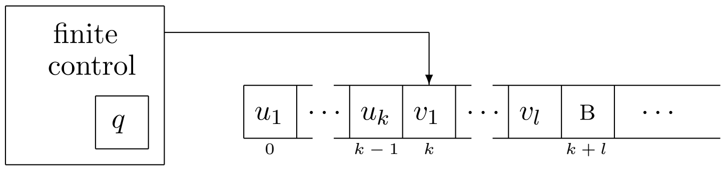


Figure 3: *A Turing machine with configuration* $uqv = u_1 \dots u_k q v_1 \dots v_l$

When making moves the Turing machine will go from one configuration to another. A configuration C_1 yields configuration C_2 if the Turing machine can legally go from C_1 to C_2 in a single step. The start configuration of a Turing machine on input ω is the configuration $q_0\omega$ which indicates that the machine is in the start state q_0 with its head at the leftmost position on the tape. In an accepting configuration the state of the configuration is a member of F.

2.2.2 Computation history

The computation history [4] for a Turing machine on an input $\omega = \omega_1\omega_2 \dots \omega_n$ is the sequence of configurations $C_0, C_1, C_2, \dots$ that the machine goes through as it processes the input. It is a complete record of the computation of the machine. C_0 is the start configuration, i.e., $C_0 = q_0\omega$, and each C_i yields C_{i+1}, which often is denoted $C_i \vdash C_{i+1}$. The symbol $\overset{*}{\vdash}$ is used to indicate that a configuration yields another configuration after several steps.

Fred Cohen [1] introduces a slightly different terminology when discussing computation histories. Let us recall that a configuration for a Turing machine is actually the setting of the three items: the state of the finite control, the tape, and the tape head. By introducing a discrete time variable, which symbolises the number of moves made by the machine, Cohen defines three functions

$$\begin{aligned} q &: N \rightarrow Q, & (1)\\ \gamma &: N \times N \rightarrow \Gamma, \text{ and} & (2)\\ P &: N \rightarrow N, & (3) \end{aligned}$$

where N is the set of natural numbers and q, γ, and P give the state, the tape contents, and the tape head position, respectively, after each move. (Observe that the symbolic notation might deviate from Fred Cohen's.) For a Turing machine on an input $\omega = \omega_1\omega_2 \ldots \omega_n$ the initial conditions are given by:

$$\begin{aligned} q(0) &= q_0, & (4)\\ \gamma(0,j) &= \begin{cases} \omega_{j+1} & ,0 \leq j < n \\ \text{B} & ,j \geq n \end{cases}, \text{ and} & (5)\\ P(0) &= 0. & (6) \end{aligned}$$

The three functions (q, γ, P) constitute the history of a Turing machine on input ω. A configuration C_i can now easily be expressed as

$$C_i = \gamma(i,0)\gamma(i,1)\cdots\gamma(i,P(i)-1)q(i)\gamma(i,P(i))\cdots\gamma(i,P(i)+k), \tag{7}$$

where k is such that $\gamma(i,P(i)+k) \neq \text{B}$ and $\gamma(i,P(i)+l) = \text{B}$ for $l > k$.

The values of q, γ, and P are determined from the initial conditions in Eqs. 4–6 and the transition function δ. From the definition of a Turing machine it follows that

$$\delta(q_i, \gamma_i) = (q_j, \gamma_j, d_j), \tag{8}$$

where $q_i, q_j \in Q, \gamma_i, \gamma_j \in \Gamma$, and $d_j \in \{L, R\}$. By introducing the functions δ_1, δ_2, and δ_3 q_j, γ_j, and d_j can be written as

$$\begin{aligned} q_j &= \delta_1(q_i, \gamma_i), & (9)\\ \gamma_j &= \delta_2(q_i, \gamma_i), \text{ and} & (10)\\ d_j &= \delta_3(q_i, \gamma_i). & (11) \end{aligned}$$

By using the transition function δ, i.e., δ_1, δ_2, and δ_3, and the expression for configuration C_i in Eq. 7, the state, tape contents, and tape head position for time $i+1$ can easily be obtained as

$$\begin{aligned}
q(i+1) &= \delta_1(q(i), \gamma(i, P(i))), && (12)\\
\gamma(i+1, j) &= \begin{cases} \delta_2(q(i), \gamma(i, P(i))) & , j = P(i) \\ \gamma(i, j) & , j \neq P(i) \end{cases}, \text{ and} && (13)\\
P(i+1) &= \begin{cases} P(i) + 1 & , \delta_3(q(i), \gamma(i, P(i))) = R \\ max(0, P(i) - 1) & , \delta_3(q(i), \gamma(i, P(i))) = L \end{cases}. && (14)
\end{aligned}$$

To summarize this subsection the different terminologies used by Sipser [4] and Cohen [1] are collected in Table 1, where the two items on each row correspond to each other.

Table 1: *Comparison of terminologies used in Ref. [4] and [1]*

Sipser [4]	Cohen [1]
Move	Time
Computation history $C_0, C_1, C_2, \ldots$	History of a machine (q, γ, P)
Configuration C_i	Situation $(q(i), \gamma(i, j), P(i)), j \geq 0)$

2.2.3 Halting

The halting of a Turing machine is a notion that seems to be defined in several ways in the literature. I will use Cohen's definition [1] which simply states that a Turing machine M halts if the situation (see Table 1) does not change with time. Formally this can be expressed as

Definition 2.2 *M halts at time t iff $\forall t' > t$*

$$\begin{aligned}
q(t) &= q(t'), \\
\gamma(t, i) &= \gamma(t', i), \forall i \geq 0, \ \textit{and} \\
P(t) &= P(t').
\end{aligned}$$

Definition 2.3 *M halts iff $\exists t \in N$ such that M halts at time t.*

Using this definition of halt, the tape head of the Turing machine defined in Definition 1 is in the leftmost position when the Turing machine has halted. Only in this position the tape head does not have to move since the instruction L leaves the tape head unchanged.

In general, it is assumed that a Turing machine halts if it accepts, i.e., if it enters an accepting state. This can be accomplished with a Turing machine in Definition 1 by making the following restriction of the transition function

$$\delta(q,\gamma) = (q,\gamma,L)\ \forall q \in F \text{ and } \forall \gamma \in \Gamma. \tag{15}$$

With this restriction the tape head of a Turing machine in an accepting state will eventually reach the leftmost position and therefore halt.

It is also possible that a Turing machine will halt even though the machine is not in an accepting state. This is perfectably accceptable and these configurations are called rejecting configurations.

Unfortunately, it is not always possible to require that a Turing machine halts on all input and this issue will be further discussed in Sections 2.3—2.5.

2.2.4 Running

Cohen also discusses the concept of a Turing machine running strings. The formal definitions are included here.

Definition 2.4 *String x runs at time t iff*

$$\begin{aligned}&x \in \Gamma^i,\ \textit{where } |x| = i > 0,\\ &q(t) = q_0,\ \textit{and}\\ &\gamma(t,P(t))\gamma(t,P(t)+1)\cdots\gamma(t,P(t)+i-1) = x.\end{aligned}$$

Definition 2.5 *String x runs iff $\exists t \in N$ such that x runs at time t.*

2.2.5 Programs

Cohen defines a Turing machine program as a finite sequence of symbols such that each symbol is a member of the legal tape symbols for the machine under consideration. TP is the set of all Turing machine programs for a specific Turing machine.

Definition 2.6 *For a given Turing machine M, the set of all Turing machine programs $TP = \{v | v \in \Gamma^i,\ \textit{where } |v| = i > 0\}$.*

Cohen continues by defining Turing machine program sets as non-empty subsets of TP.

Definition 2.7 *For a given Turing machine M, V is a Turing machine program set iff $V \subset TP$ and $V \neq \emptyset$.*

Finally, the set TS of all Turing machine program sets can be defined.

Definition 2.8 *For a given Turing machine M, the set of all Turing machine program sets* $TS = \{V | V \subset TP, V \neq \emptyset\}$.

2.2.6 Evolution

Let M be a Turing machine, V a program set for M, and v a string in V, then the evolution of v to another member of V for M is denoted by Cohen as $v \overset{M}{\Rightarrow} V$ and simply means that if v runs at a certain time then another member of V will be written to the tape, not overwriting v, at some later time.

Definition 2.9 *For a given Turing machine M, for* $V \in TS$, *and* $v \in V$, $v \overset{M}{\Rightarrow} V$ *iff* $\forall t$ *v runs at* $t \Rightarrow \exists v' \in V$, $\exists t' > t$, *and* $\exists j' \geq 0$ *such that*

$$\begin{array}{l} j' + |v'| \leq P(t) \text{ or } P(t) + |v| \leq j', \\ \gamma(t', j') \cdots \gamma(t', j' + |v'| - 1) = v', \text{ and} \\ \exists t'' \text{ such that } t < t'' < t' \text{ and } P(t'') \in \{j', \ldots, j' + |v'| - 1\}. \end{array}$$

2.2.7 Transition diagrams

The transitions of a Turing machine can be represented pictorially in a so called transition diagram. A transition diagram consists of nodes corresponding to the states of the Turing machine. An arc from state q to state p is labeled by an item of the form X/YD, where X and Y are tape symbols, and D is a direction, either L or R. That is, whenever $\delta(q, X) = (p, Y, D)$, the arc from q to p is labeled X/YD (see Figure 4). However, in the diagrams the direction D is represented pictorially by $\leftarrow$ for left and $\rightarrow$ for right.

The start state is represented by the word "start" and an arrow entering that state. Accepting states are indicated by double circles. Thus, the only information about the Turing machine one cannot read directly from the transition diagram is the symbol used for the blank. It is assumed that that symbol is B.

Example 2.1 *An illustration (see Figure 4) of a transition diagram will be given for the Turing machine* $M = (\{q_0, q_1, q_2\}, \{0, 1\}, \{0, 1, B\}, \delta, q_0, B, q_1)$, *where the transition function is given by*

$$\begin{array}{rcl} \delta(q_0, 0) & = & (q_1, 0, R), \\ \delta(q_0, 1) & = & (q_0, 1, R), \\ \delta(q_0, B) & = & (q_2, B, L), \\ \delta(q_1, X) & = & (q_1, X, L), \ X \in \Gamma \\ \delta(q_2, X) & = & (q_2, X, L), \ X \in \Gamma. \end{array}$$

In Figure 4 the symbol $*$ denotes all tape symbols.

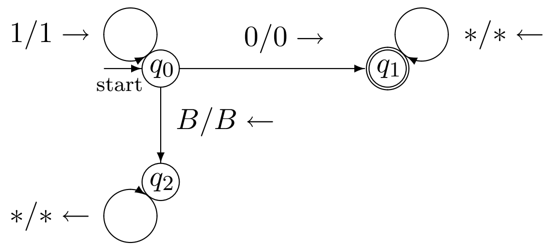


Figure 4: *Transition diagram for Turing machine M*

2.3 Languages and problems

Is is now possible to make a formal definition of a language of a Turing machine. By using the definition in Section 2.2 of a Turing machine, including the restriction of the transition function in Equation 15, the following definition can be made.

Definition 2.10 *Let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ be a Turing machine. Then $L(M)$ is the set of strings ω in Σ^* such that $q_0\omega \overset{*}{\vdash} p\alpha$ for some state p in F and any tape string α.*

Using this definition it is easily verified that the language of the Turing machine M in Example 2.1 is

$$L(M) = \{\omega | \omega \text{ has at least one zero}\}.$$

The set of languages that can be accepted using a Turing machine is often called the *recursively enumerable languages* or RE languages. In Sipser [4] this set of languages is simply called Turing recognizable.

In automata theory, a *problem* is the question of deciding whether a given string is a member of some particular language. More precisely, if Σ is an alphabet, and L is a language over Σ, then the problem L is:

- Given a string ω in Σ^*, decide whether or not ω is in L.

Languages and problems are really the same thing. Which term to use depends on the point of view. If strings are considered only for their own sake then the set of strings is thought of as a language. If, on the other hand, semantics are assigned to the strings, e.g., think of strings as coding graphs, logical expressions, or even integers, then what the string represents is more important than the string itself, and in those cases the set of strings is thought of as a problem.

2.4 Decidability

Unfortunately, it is not always possible to require that a Turing machine halts even if it does not accept. Those languages with Turing machines that do halt eventually, regardless of whether or not they accept, are called *recursive*. The language of the Turing machine in Example 2.1 is an example of an recursive language. Turing machines that always halt, regardless of whether or not they accept, are a good model of an algorithm. If an algorithm to solve a given problem exists, then the problem is decidable.

2.4.1 The halting problem

One might think that undecidable problems are rare, but in fact they are not. The halting problem is a well-known example of an undecidable problem and it can be formulated in the following way (TM stands for Turing machine).

Theorem 2.1 *Let $A_{TM} = \{\langle M, \omega\rangle | M \text{ is a } TM \text{ and } M \text{ accepts } \omega\}$. Then A_{TM} is undecidable.*

Proof 2.1 *In a proof by contradiction it will be assumed that A_{TM} is decidable. Suppose that H is a decider for A_{TM}, i.e.,*

$$H(\langle M, \omega\rangle) = \begin{cases} \textit{accept} & \textit{if } M \textit{ accepts } \omega \\ \textit{reject} & \textit{if } M \textit{ does not accept } \omega. \end{cases}$$

A new Turing machine D with H as a subroutine will now be constructed.

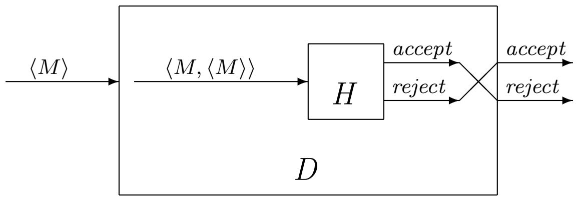


Figure 5: *Turing machine D for proving the halting problem undecidable*

The input to D is a description of a Turing machine M. This information is sent to Turing machine H which determines what M does when the input to M is its own description. Once D has determined this information, it does the opposite, i.e., it rejects if M accepts and accepts if M does not accept:

$$D(\langle M\rangle) = \begin{cases} \textit{accept} & \textit{if } M \textit{ does not accept } \langle M\rangle \\ \textit{reject} & \textit{if } M \textit{ accepts } \langle M\rangle. \end{cases}$$

In the case when D is run with its own description the following is obtained:

$$D(\langle D\rangle) = \begin{cases} \textit{accept} & \textit{if } D \textit{ does not accept } \langle D\rangle \\ \textit{reject} & \textit{if } D \textit{ accepts } \langle D\rangle. \end{cases}$$

Here a case of obvious contradiction is obtained. Thus neither D nor H can exist. Q.E.D.

The reason for the name "halting problem" can be traced back to the discussion in Section 2.2.3 of halting. A Turing machine defined as in Section 2.2 halts if it either accepts or rejects its input. However, it is not always possible to require that a Turing machine halts on all input. For a further discussion see Section 2.5.

2.4.2 The universal Turing machine

Although the language A_{TM} is not recursive it is recursively enumerable since there is a Turing machine that recognizes it. The Turing machine U in Figure 6 recognizes A_{TM}.

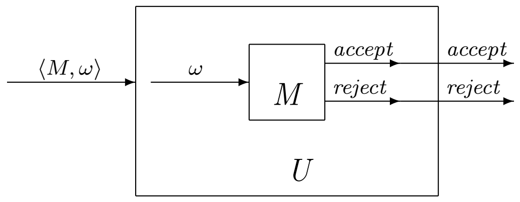


Figure 6: *The universal Turing machine U*

The Turing machine U is an example of the universal Turing machine first proposed by Turing. The machine is called universal because of its capability to simulate all other Turing machines from their descriptions. As will be seen below in Section 3, the concept of the universal Turing machine was essential in the development of stored-program computers.

One should note that the universal Turing machine loops on input $\langle M, \omega \rangle$ if M loops on ω, which is why this machine does not decide A_{TM}. The language A_{TM} is sometimes called the universal language.

2.5 Reducibility

Reducibility is the primary method for proving that problems are undecidable or decidable. A reduction is a way of converting one problem into another problem in such a way that the solution to the second problem can be used to solve the first problem. Knowing a problem like A_{TM} being undecidable can be used to prove other problems being undecidable. If a problem A is undecidable and reducible to another problem B then B is undecidable. A similar reasoning can be used for decidable problems. If a problem A is reducible to another problem B and B is decidable then A is also decidable.

The process of using reducibility to prove problems undecidable will be illustrated below in a few examples. The undecidability of A_{TM}, the problem of determining whether a Turing machine accepts a given input, has already been established. A related problem is $HALT_{TM}$, the problem of determining whether a Turing machine halts (by accepting or rejecting) on a given input. The undecidability of A_{TM} will be used to prove the undecidability of $HALT_{TM}$ by reducing A_{TM} to $HALT_{TM}$.

Theorem 2.2 *Let $HALT_{TM} = \{\langle M, \omega\rangle | M$ is a TM and M halts on input $\omega\}$. Then $HALT_{TM}$ is undecidable.*

Proof 2.2 *In a proof by contradiction it will be assumed that $HALT_{TM}$ is decidable and this assumption will be used to show that A_{TM} is decidable, contradicting Theorem 2.1. The key idea is to show that A_{TM} is reducible to $HALT_{TM}$.*

Suppose that R is a decider for $HALT_{TM}$, i.e.,

$$R(\langle M, \omega\rangle) = \begin{cases} \textit{accept} & \textit{if } M \textit{ halts on input } \omega \\ \textit{reject} & \textit{if } M \textit{ does not halt on input } \omega. \end{cases}$$

A new Turing machine S to decide A_{TM}, with R as a subroutine, will now be constructed.

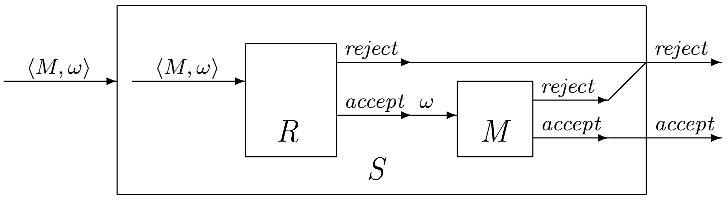


Figure 7: *Turing machine S for deciding A_{TM}*

Clearly, if R decides $HALT_{TM}$, then S decides A_{TM}. Because A_{TM} is undecidable, $HALT_{TM}$ also must be undecidable, i.e., R does not exist. Q.E.D.

Theorem 2.3 *Let $E_{TM} = \{\langle M\rangle | M$ is a TM and $L(M) = \emptyset\}$. Then E_{TM} is undecidable.*

Proof 2.3 *In a proof by contradiction it will be assumed that E_{TM} is decidable and this assumption will be used to show that A_{TM} is decidable, contradicting Theorem 2.1. The key idea is to show that A_{TM} is reducible to E_{TM}.*

Suppose that R is a decider for E_{TM}, i.e.,

$$R(\langle M\rangle) = \begin{cases} \textit{accept} & \textit{if } L(M) = \emptyset \\ \textit{reject} & \textit{if } L(M) \neq \emptyset. \end{cases}$$

A new Turing machine S to decide A_{TM}, with R as a subroutine, will now be constructed.

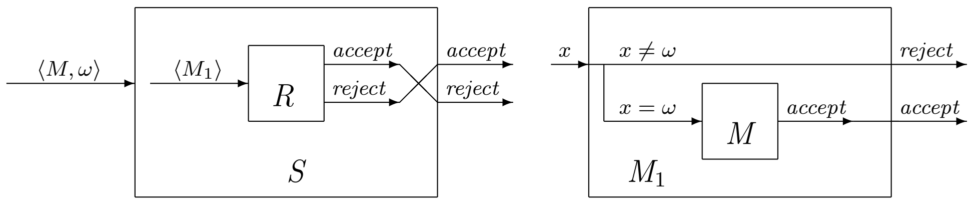


Figure 8: *Turing machine S for deciding A_{TM}*

The Turing machine M_1 has the string ω as part of its description. Obviously, $L(M_1)$ is nonempty only when M accepts ω as input. Note that S must be able to compute a description of M_1 from a description of M and ω. It is able to do so because it needs only add extra states to M that perform the $x = \omega$ test.

Clearly, if R decides E_{TM}, then S decides A_{TM}. Because A_{TM} is undecidable, E_{TM} also must be undecidable, i.e., R does not exist. Q.E.D.

Theorem 2.4 *Let $REGULAR_{TM} = \{\langle M \rangle | M$ is a TM and $L(M)$ is a regular language$\}$. Then $REGULAR_{TM}$ is undecidable.*

Proof 2.4 *In a proof by contradiction it will be assumed that $REGULAR_{TM}$ is decidable and this assumption will be used to show that A_{TM} is decidable, contradicting Theorem 2.1. The key idea is to show that A_{TM} is reducible to $REGULAR_{TM}$.*

Suppose that R is a decider for $REGULAR_{TM}$, i.e.,

$$R(\langle M \rangle) = \begin{cases} \textit{accept} & \textit{if } L(M) \textit{ is a regular language} \\ \textit{reject} & \textit{if } L(M) \textit{ is not a regular language.} \end{cases}$$

A new Turing machine S to decide A_{TM}, with R as a subroutine, will now be constructed.

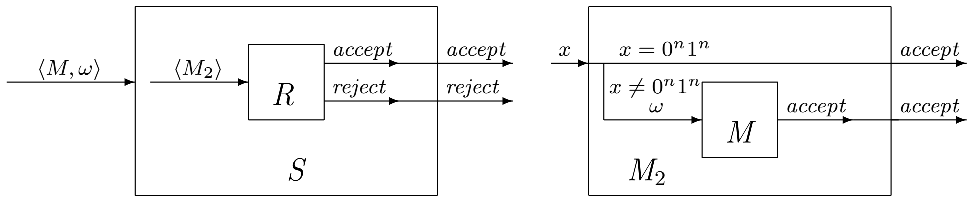


Figure 9: *Turing machine S for deciding A_{TM}*

The Turing machine M_2 recognizes the nonregular language $\{0^n1^n | n \geq 0\}$ if M does not accept ω and the regular language Σ^* if M accepts ω. M_2 works by automatically accepting all strings in $\{0^n1^n | n \geq 0\}$. In addition, if M accepts ω, M_2 accepts all other strings.

Clearly, if R decides $REGULAR_{TM}$, then S decides A_{TM}. Because A_{TM} is undecidable, $REGULAR_{TM}$ also must be undecidable, i.e., R does not exist. Q.E.D.

So far, in the theorems given above, the proofs have involved a reduction from A_{TM}. Sometimes reducing from another undecidable language, such as E_{TM}, is more convenient. This will be demonstrated in the next theorem.

Theorem 2.5 *Let $EQ_{TM} = \{\langle M_1, M_2\rangle | M_1$ and M_2 are TMs and $L(M_1) = L(M_2)\}$. Then EQ_{TM} is undecidable.*

Proof 2.5 *In a proof by contradiction it will be assumed that EQ_{TM} is decidable and this assumption will be used to show that E_{TM} is decidable, contradicting Theorem 2.3. The key idea is to show that E_{TM} is reducible to EQ_{TM}.*

Suppose that R is a decider for EQ_{TM}, i.e.,

$$R(\langle M_1, M_2\rangle) = \begin{cases} \textit{accept} & \textit{if } L(M_1) = L(M_2) \\ \textit{reject} & \textit{if } L(M_1) \neq L(M_2). \end{cases}$$

A new Turing machine S to decide E_{TM}, with R as a subroutine, will now be constructed.

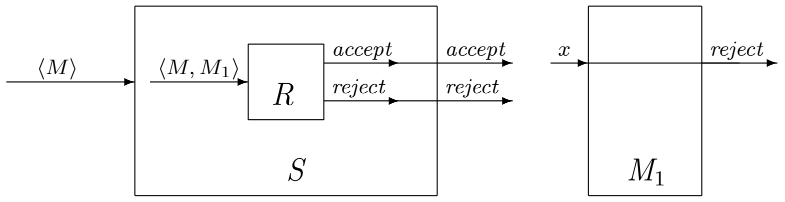


Figure 10: *Turing machine S for deciding E_{TM}*

The Turing machine M_1 simply rejects all input and therefore $L(M_1)$ is empty. Surely, the language of a Turing machine M is empty only if it is equal to the language of M_1.

Clearly, if R decides EQ_{TM}, then S decides E_{TM}. Because E_{TM} is undecidable, EQ_{TM} also must be undecidable, i.e., R does not exist. Q.E.D.

2.6 The recursion theorem

The recursion theorem plays an important role in advanced work in computability theory. It has connections to computer viruses and this is the reason for mentioning it in this paper. To get a feeling for the recursion theorem it is appropriate to consider a paradox that arises in the study of life. The paradox can be summarized as follows:

- Living things are machines.
- Living things can self-reproduce.
- Machines cannot self-reproduce.

The first statement follows from modern biology. The second statement is obvious. What about the third statement? To resolve the paradox, it has to be concluded that the third statement is incorrect. Making machines that reproduce themselves is possible, and the recursion theorem demonstrates how. But first another theorem has to be proven.

Theorem 2.6 *There is a computable function q: $\Sigma^* \rightarrow \Sigma^*$, where for any string ω, $q(\omega)$ is the description of a Turing machine P_ω that prints out ω and then halts.*

Proof 2.6 *The following Turing machine Q computes $q(\omega)$.*

```
$Q =$ "On input string $\omega$.
    1. Construct the following Turing machine $P_\omega$:
       $P_\omega =$ "On any input:
           1. Erase input.
           2. Write $\omega$ on the tape.
           3. Halt."
    2. Output $\langle P_\omega \rangle$."
```

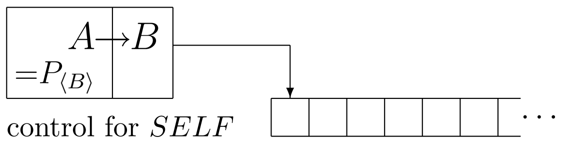
Q.E.D.

Using Theorem 2.6 a Turing machine that ignores its input and prints out a copy of its own description can be constructed. This machine will be called $SELF$. The description of $SELF$ facilitates the understanding of the recursion theorem. The Turing machine $SELF$ is divided into two parts A and B. A and B can be thought of as two separate procedures that go together to make up $SELF$. The purpose of $SELF$ is to print out $\langle SELF \rangle = \langle AB \rangle$. Part A runs first and its task is to print out a description of B. When A is finished control passes over to B whose task is to print out a description of A. The jobs of A and B are similar but they have to be carried out differently for not ending up in circular definitions.

Since A should print out a description of B, the machine $P_{\langle B \rangle}$ will be used for A. The description of A is $\langle A \rangle = \langle P_{\langle B \rangle} \rangle = q(\langle B \rangle)$ and it depends on having a description of B. So the description of A cannot be completed until B has been constructed.

When part B starts running $\langle B \rangle$ is left on the tape. By obtaining it and applying q to it then $\langle A \rangle$ can be obtained since $\langle A \rangle = q(\langle B \rangle)$. By adding

$\langle A \rangle$ to the front of the tape, the tape finally contains $\langle AB \rangle = \langle SELF \rangle$. In summary:



$B =$ "On input $\langle M \rangle$

1. Compute $q(\langle M \rangle)$

2. Combine the result with $\langle M \rangle$

3. Print this description and halt."

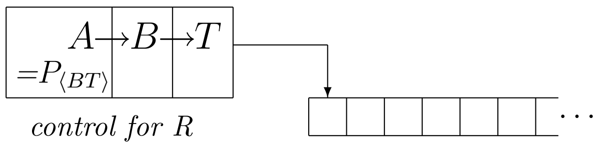
Theorem 2.7 Recursion Theorem *Let T be a Turing machine that computes a function t: $\Sigma^* \times \Sigma^* \to \Sigma^*$. There is a Turing machine R that computes a function r: $\Sigma^* \to \Sigma^*$, where for every ω, $r(\omega) = t(\langle R \rangle, \omega)$.*

The statement of this theorem may seem more technical than it is. To make a Turing machine that can obtain its own description and then compute with it, there is only need to make a machine, called T in the statement, that takes an extra input that receives the description of the machine. Then the recursion theorem produces a new machine R, which operates as T but with R's description filled in automatically.

Proof 2.7 *Construct a Turing machine R in three parts, A, B, and T, where T is given by the statement of the theorem.*

- *A is the Turing machine $P_{\langle BT \rangle}$ described by $q(\langle BT \rangle)$. After A runs, the tape contains $\langle BT \rangle$.*

- *B is a Turing machine that examines its tape and applies q to its contents. The result is $q(\langle BT \rangle) = \langle A \rangle$, the description of A. Having the descriptions of A and BT, i.e., R, B then combines A, B, and T into a single machine, writes its description on the tape, and passes control to T.*



Q.E.D.

The recursion theorem states that Turing machines have the capability to obtain their own description and then go on to compute with it. Further, the recursion theorem is a handy tool for solving certain problems concerning the theory of algorithms. For more details and examples where the recursion theorem is used see Ref. [4].

3 Computers

For most people today, surrounded by computers in daycares, schools, and working places, it is quite obvious what a computer is: a digital computer with internally stored modifiable programs. However, the meaning of the word has changed in time. In the 1930s and 1940s (during Alan Turing's scientifically active period) a computer meant a person doing calculations. Also the idea of storing programs in just the same way as data was not at all obvious at that time. Builders of large electromechanical calculaters put the program on cards or on a roll of punched paper, i.e., the machinery should do the arithmetics and the instructions should be coded in some other form. Even when turning to electronics (ENIAC in 1943), programs were still thought of something quite different from numbers. A breakthrough came in 1945 with the EDVAC report by John von Neumann where the basic elements of the stored program concept were introduced to the industry. This concept had been discussed with the ENIAC engineers Eckert, Mauchly and others in connection with their plans for a successor machine to the ENIAC.

Beeing a mathematician John von Neumann had become fascinated by non-linear partial differential equations in the 1930s. The phenomena described by these equations are baffling analytically and numerical work seemed to him the most promising way to obtain a feeling for the behaviour of these systems. This was a driving force for studying new possibilities of computation on electronic machines. Beeing a mathematician he most surely also was familiar with Alan Turing's work involving the concepts of logical design and the universal machine. However, whether von Neumann applied Turing's ideas to the design of a computer is questionable. John von Neumann's logical design of a computer became the prototype of most of its successors—the von Neumann architecture (see Figure 11).

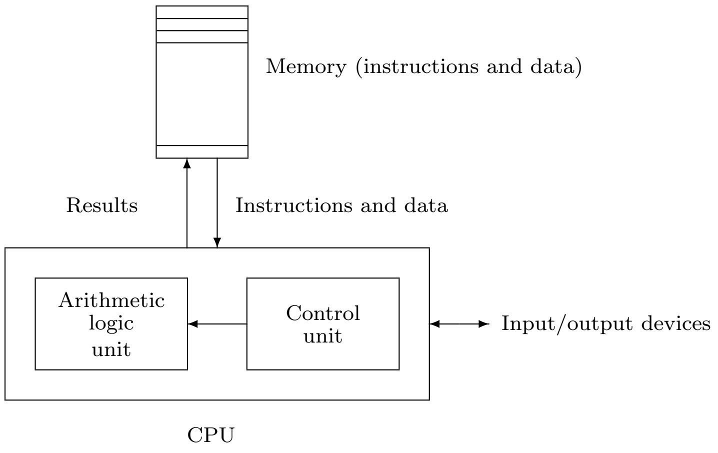


Figure 11: *The von Neumann architecture*

Almost at the same time, in 1945, Alan Turing presented a report, independent of the EDVAC proposel, concerning the design of a modern computer. This work was inspired by his own 1936 concept of the universal machine, his experiences during the war of the speed and reliability of electronics, and the realization of the inefficiency in designing different machines for different logical processes. Turing's 1945 conception of a computer was not tied to numbers but for the logical manipulation of symbols of any kind. From the start he stressed that the universal machine should be able to switch at a moments notice from arithmetic to the algebra of group theory, to chess playing, or to data processing. Further, he saw immediately the first ideas of programming structure and languages. In addition, he was spurred by the idea that the universal machine should be able to acquire and exhibit the faculties of the human mind. In June 1948, in Manchester, the world's first practical realization of Turing's computer principle was demonstrated.

4 Computer viruses

On November 3, 1983, the first computer virus was conceived as an experiment to be presented at a weekly seminar on computer security. The concept was introduced in the seminar by Fred Cohen and in 1986 he presented his PhD Thesis [1] covering this subject. However, computer viruses can trace their ancestor to John von Neumann's studies of self-replicating mathematical automata in the 1940s.

For an informal first introduction of the concept of a computer virus the analogy from the biological world is useful. A biological virus attach itself to a cell and inserts a bit of biological "code" into the cell so that the virus will be reproduced and spread by that particular cell. In a computer the cells are represented by executable files, i.e., compiled programs in machine code. What computer viruses do is to insert themselves into executable files. When such a file later on is running the virus can do annoying things on the computer and it can spread itself to other executable files.

In Section 4.1 below a formal definition of a virus in connection to Turing machines will be given. For this definition the concept of viral sets is essential.

4.1 Viral sets

By using the definitions in Section 2, the concept of the viral set can be defined.

4.1.1 Definition

Definition 4.1 *The viral set* $VS = \{(M,V) | M \ is \ a \ Turing \ machine, \ V \in TS \ for \ M, \ and \ \forall v \in V, \ v \overset{M}{\Rightarrow} V\}$.

Another definition that will be used frequently below is the definition of viral sets with respect to Turing machines.

Definition 4.2 *V is a viral set with respect to a Turing machine M iff (M,V)* $\in$ *VS.*

A virus can now be defined as a member of a viral set, V, with respect to a Turing machine M.

Definition 4.3 *v is a virus with respect to a Turing machine M, if* $v \in V$ *and (M,V)* $\in$ *VS.*

The following quotation from Cohen's thesis [1] is illustrative.

> The sequence of tape symbols we call "viruses" is a function of the machine on which they are to be interpreted. In particular, we may expect that a given sequence of symbols may be a "virus" when interpreted by one TM and not a "virus" when interpreted by another TM.

According to Cohen [1] several previous attempts at definition failed because the idea of a singleton virus makes the understanding of evolution of viruses very difficult. And evolution is indeed a central concept.

4.1.2 Evolution of viruses

A number of definitions concerning evolution of viruses will now follow.

Definition 4.4 *v evolves into v' for a Turing machine M iff v and v' are viruses and* $v \overset{M}{\Rightarrow} \{v'\}$.

Definition 4.5 *v' is evolved from v for a Turing machine M iff v evolves into v' for M.*

Definition 4.6 *v' is an evolution of v for a Turing machine M iff* $(M,V) \in VS$ *and* $\exists V' \subset V$ *such that* $\forall v_k \in V', \ v_k \overset{M}{\Rightarrow} v_{k+1}$ *and* $\exists l, m \in N$ *such that* $l < m, \ v_l = v$, *and* $v_m = v'$.

The definition above is deviating slightly from the formulation in Cohen's thesis [1]. However, the content should be the same. The following quotation from Cohen's thesis [1] summarizes the definitions above.

> The "viral set" embodies evolution by allowing elements of such a set to produce other elements of that set as a result of computation. So long as each "virus" in a "viral set" produces some elements of that "viral set" on some part of the tape outside of the original "virus", the set is considered "viral". Thus "evolution" may be described as the production of one element of a "viral set" from another element of that set.

4.1.3 Basic theorems

Cohen gives a number of basic theorems concerning viral sets and these will be repeated here without proofs. For proofs see Cohen's thesis [1].

Theorem 4.1 *Any union of viral sets is also a viral set, i.e., if* $(M, V_1) \in VS$ *and* $(M, V_2) \in VS$ *then* $(M, V_1 \cup V_2) \in VS$.

From Theorem 4.1 it follows that the largest viral set with respect to any machine is the union of all viral sets with respect to that machine.

For the next theorems the concept of smallest viral set has to be defined.

Definition 4.7 *A smallest viral set is a viral set of which no proper subset is a viral set with respect to the given machine. There may be many such sets for a given machine.*

Theorem 4.2 *There is a machine for which the smallest viral set is a singleton set, and that the minimal viral set is therefore singleton.*

It can also be shown that any sequence which duplicates itself is a virus with respect to the machine on which it is self duplicating. These are examples of singleton viral sets. In fact the smallest viral sets come in all sizes as the next theorem states.

Theorem 4.3 *For any finite integer i there is a machine such that there is a smallest viral set with i elements.*

4.2 Subroutines

In order to simplify the presentation of a number of proofs concerning the computability aspects of viruses and viral detection some subroutines will be used and they will be defined here. The subroutines will be expressed as transition diagrams and the notation is the same as the one in Example 2.1 in Section 2.2.7.

The "halt" subroutine will allow halting of a Turing machine in any given state q_n (see Figure 12).

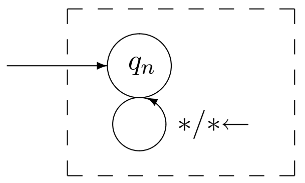


Figure 12: *The "halt" subroutine*

The "right till x" (or $R(x)$) subroutine will allow a Turing machine to increment the position of the tape head until such a position is reached that the symbol x is in front of the tape head (see Figure 13).

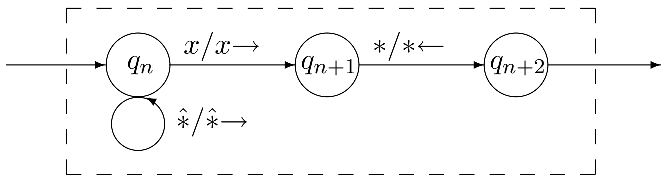


Figure 13: *The "right till x" subroutine ($\hat{*}$ denotes all symbols except x)*

The "left till x" (or $L(x)$) subroutine is just like the $R(x)$ subroutine except that the tape head is moving left rather than right (see Figure 14).

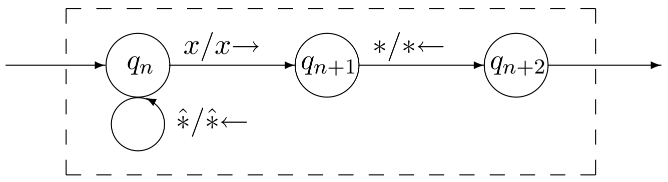


Figure 14: *The "left till x" subroutine ($\hat{*}$ denotes all symbols except x)*

The next subroutine to be discussed, the "copy from x till y after z" (or $CPY(x, y, z)$), is essential when discussing viruses. It is more complex than the other subroutines discussed since the number of states depends on the number of input symbols for the machine under consideration. See the loop starting and ending in state q_{n+13} in Figure 15. If all input symbols get a unique number then $7^{\hat{*}}$ symbolizes 7 to the power of that unique number. Another complexity is that the copy subroutine uses a "left of tape marker" (N) and a "right of tape marker" (M). These symbols should not be used as input symbols, i.e., $N, M \in \Gamma$ but $N, M \notin \Sigma$. The left of tape marker N is used to mark where the next symbol should be read and the right of tape marker M is used to mark where the next symbol should be written.

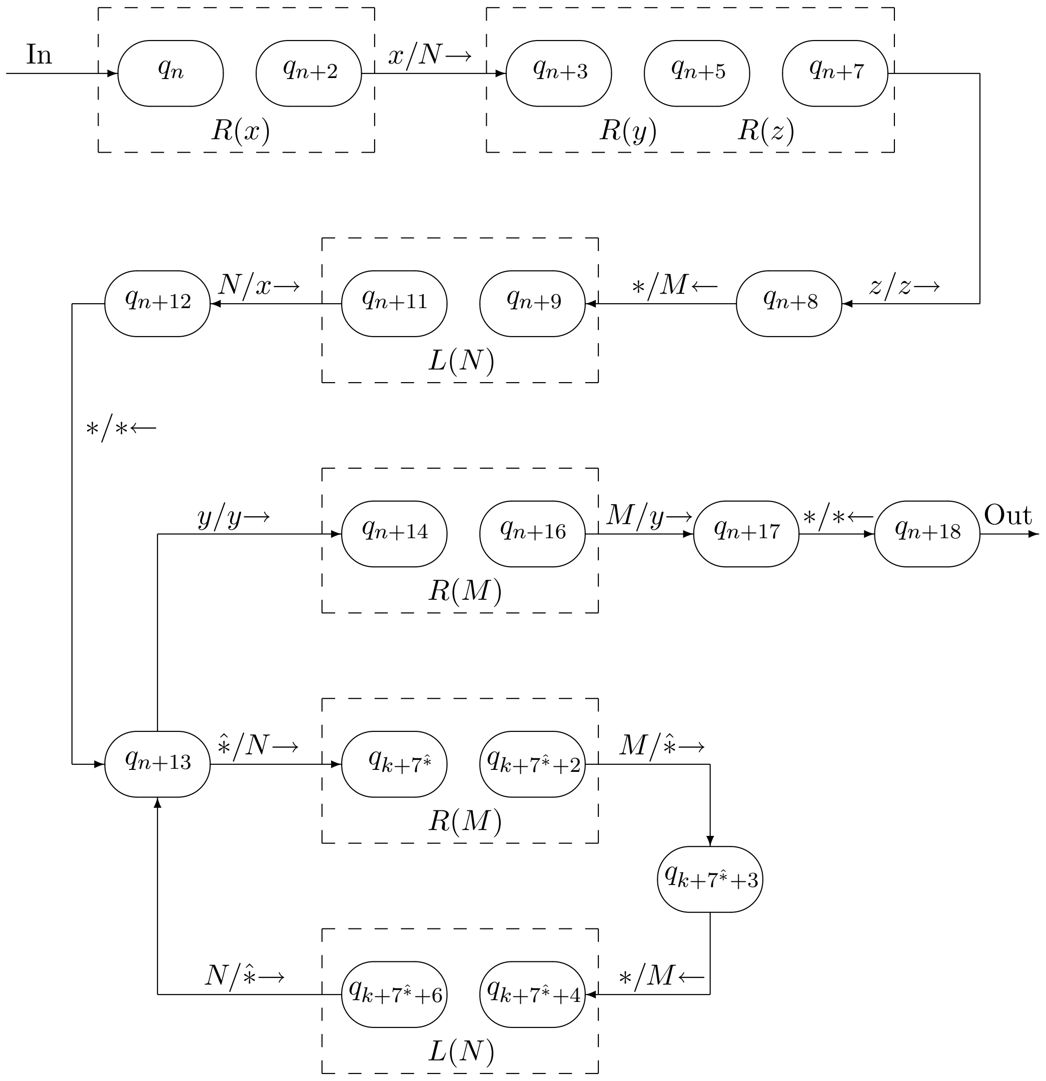
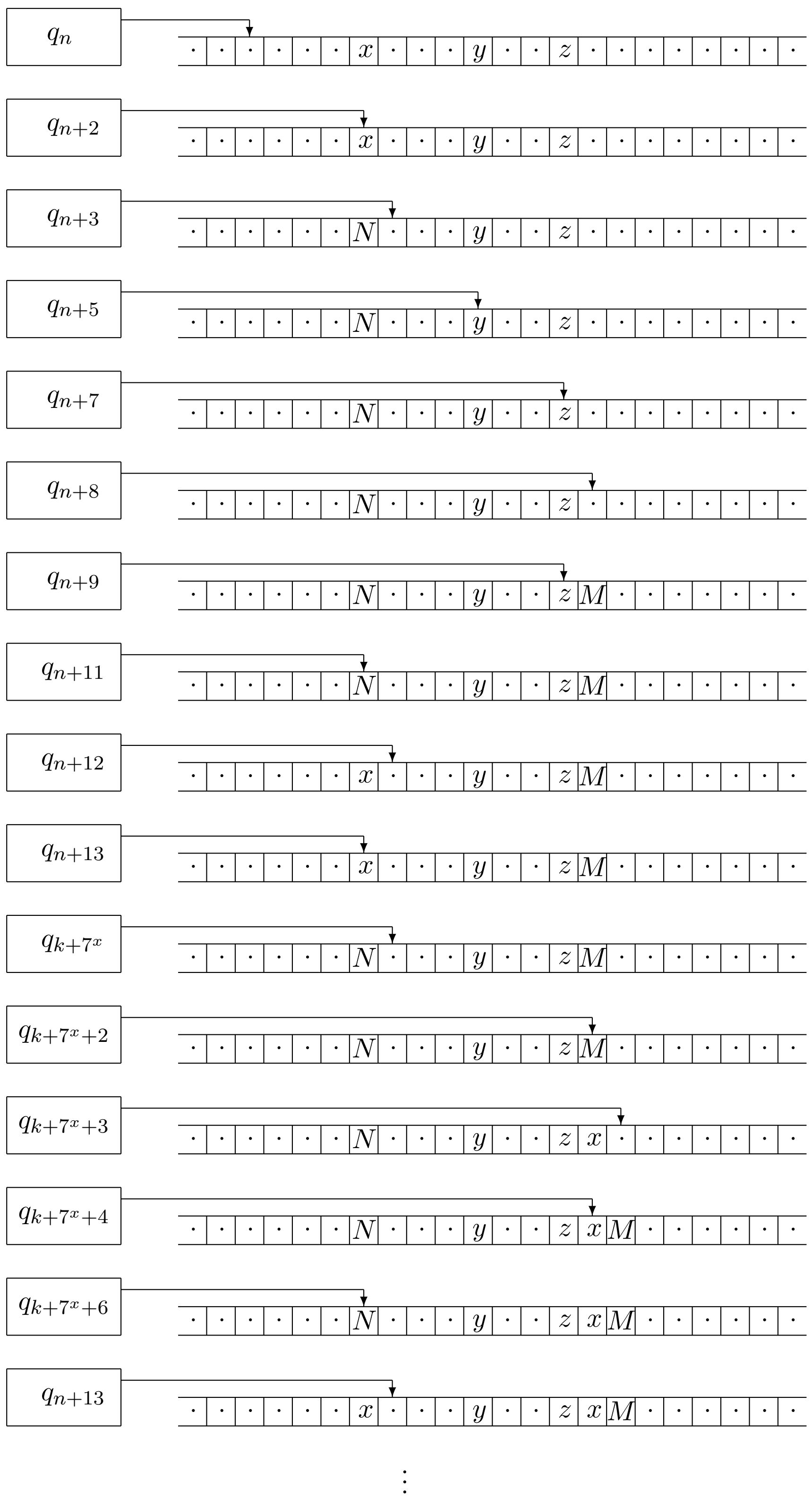


Figure 15: *The "copy from x till y after z" subroutine*
($\hat{*}$ denotes all symbols except y and $k \geq n + 19$)

As can be seen from Figure 15, the copy subroutine makes extensive use of the left and right subroutines. This makes sense since the tape head has to move back and forth in order to take one symbol and copy it. The workings of the copy subroutine may best be described using pictures and in Figure 16 a series of snapshots of a Turing machine running the subroutine is given.


q_n
q_{n+2}
q_{n+3}
q_{n+5}
q_{n+7}
q_{n+8}
q_{n+9}
q_{n+11}
q_{n+12}
q_{n+13}
q_{k+7^x}
q_{k+7^x+2}
q_{k+7^x+3}
q_{k+7^x+4}
q_{k+7^x+6}
q_{n+13}
x
y
z
N
M

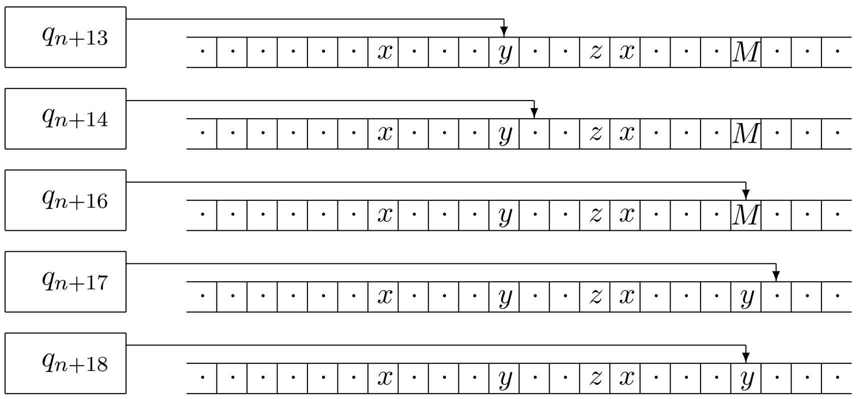


Figure 16: *Running of the CPY(x,y,z) subroutine*

The last subroutine to be discussed is the "generate v" (or $generate(v)$) subroutine. This subroutine will allow a Turing machine to generate a string v starting from the current tape head position (see Figure 17).

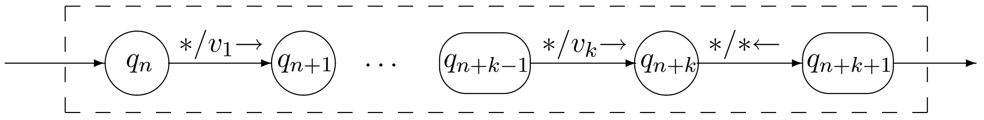


Figure 17: *The "generate v" subroutine ($v = v_1v_2\cdots v_k$)*

When the subroutines above will be used in Section 4.3 only the starting and ending states will be shown and these will be numbered consecutively, for instance by q_0 and q_1, respectively. It has to be assumed that the states in between these states are numbered differently.

4.3 Computability aspects of viruses and viral detection

In the following three subsections three issues concerning the power of viruses will be explored. First the "decidability" issue will be addressed. This concerns the question of whether there is a Turing machine capable of determining in finite time whether or not a given sequence for a given Turing machine is a virus. The second issue that will be addressed is the "evolution" issue. This concerns the question of whether there is a Turing machine capable of determining in finite time whether or not a given virus for a given Turing machine generates another given virus for that machine. The third and last issue that will be addressed is the "computability" issue. It will be shown that any number that can be computed by a Turing machine can be evolved by a virus, and that therefore, viruses are at least as powerful as Turing machines as means for computation.

4.3.1 Decidability

It will now be shown that it is undecidable whether or not a given (M,V) pair belongs to the viral set. The proof below will follow the ideas in Section 2.5 and in the proof of Theorem 6 in Cohen's thesis [1]. But the presentation below will be somewhat simplified.

Theorem 4.4 *Let* $VS' = \{(M, \{\omega\}) | (M, \{\omega\}) \in VS\}$*. Then* VS' *is undecidable.*

Proof 4.4 *In a proof by contradiction it will be assumed that* VS' *is decidable and this assumption will be used to show that* $HALT_{TM}$ *is decidable, contradicting Theorem 2.2. The key idea is to show that* $HALT_{TM}$ *is reducible to* VS'*.*

Suppose that D is a decider for VS'*, i.e.,*

$$D(\langle M, \omega \rangle) = \begin{cases} \textit{accept} & \textit{if } (M, \{\omega\}) \in VS \\ \textit{reject} & \textit{if } (M, \{\omega\}) \notin VS. \end{cases}$$

A new Turing machine H to decide $HALT_{TM}$*, with D as a subroutine, will now be constructed (see Figure 18).*

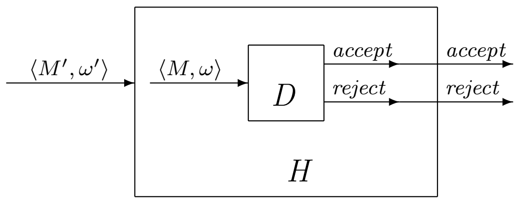


Figure 18: *Turing machine H for deciding* $HALT_{TM}$

The input to D is a Turing machine M, which is defined in Figure 19, and a string $\omega = Ll\omega' rR$*. Here it is assumed that* $L, l, r, R \notin \Sigma_{M'}$*, that* M' *goes to state* q_x *if and only if* M' *halts, and that* $\begin{tikzcd} q'_n \arrow[loop right] \end{tikzcd}$ $l/l\rightarrow$, $\forall q'_n \in Q_{M'}$*.*

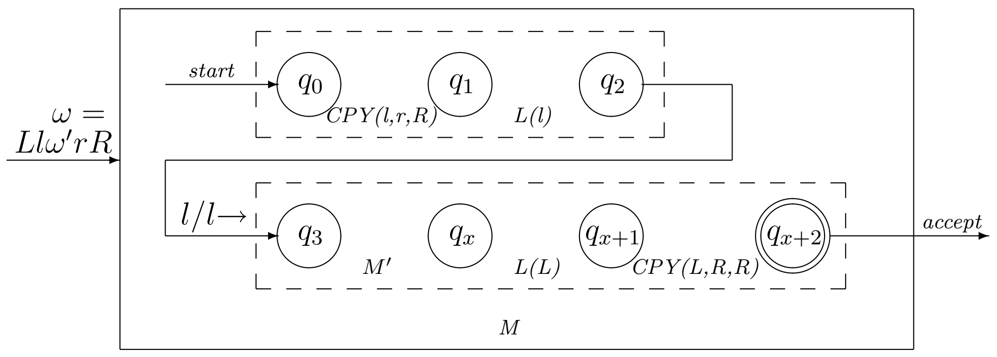


Figure 19: *Turing machine M and input string ω used in Figure 18*

What M simply does is to copy ω' from inside of ω, simulate the execution of M' on the copy of ω', and if M' halts replicate ω. Thus M replicates ω if and only if M' halts on input ω'. This will work since the assumptions made above will not allow M' to corrupt the string ω and L is only found in string ω. In order to increase the understanding of the workings of M, the running of ω on M is given in Figure 20.

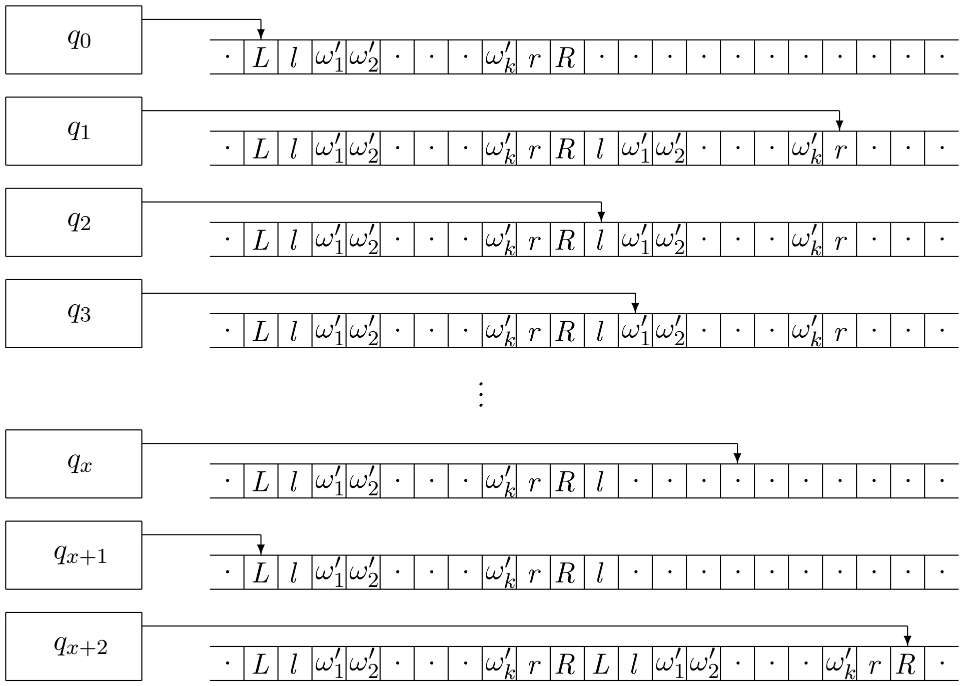


Figure 20: *Running of $Ll\omega' rR$ on M*

Q.E.D.

Now when having proved *VS'* undecidable it is a simple task to prove that *VS* is undecidable by simply reducing *VS'* to *VS*.

Theorem 4.5 *VS is undecidable.*

Proof 4.5 *In a proof by contradiction it will be assumed that VS is decidable and this assumption will be used to show that VS' is decidable, contradicting Theorem 4.4. The key idea is to show that VS' is reducible to VS.*

Suppose that D is a decider for VS, i.e.,

$$D(\langle M, V\rangle) = \begin{cases} \textit{accept} & \textit{if } (M, V) \in VS \\ \textit{reject} & \textit{if } (M, V) \notin VS. \end{cases}$$

A new Turing machine V to decide VS', with D as a subroutine, will now be constructed (see Figure 21).

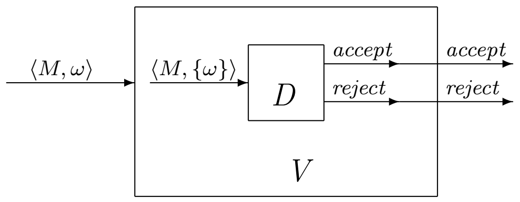


Figure 21: *Turing machine V for deciding VS'*

Q.E.D.

4.3.2 Evolution

It will now be shown that it is undecidable whether or not a given virus v evolves into another given virus v' for a Turing machine M. The proof will be similar to the proof of Theorem 4.4.

Theorem 4.6 *Let $EV_{TM} = \{(M, v, v') | v$ evolves into v' for Turing machine $M\}$. Then EV_{TM} is undecidable.*

Proof 4.6 *In a proof by contradiction it will be assumed that EV_{TM} is decidable and this assumption will be used to show that $HALT_{TM}$ is decidable, contradicting Theorem 2.2. The key idea is to show that $HALT_{TM}$ is reducible to EV_{TM}.*

Suppose that D is a decider for EV_{TM}, i.e.,

$$D(\langle M, v, v' \rangle) = \begin{cases} \textit{accept} & \textit{if } v \textit{ and } v' \textit{ are viruses and } v \overset{M}{\Rightarrow} \{v'\} \\ \textit{reject} & \textit{otherwise.} \end{cases}$$

A new Turing machine H to decide $HALT_{TM}$, with D as a subroutine, will now be constructed (see Figure 22).

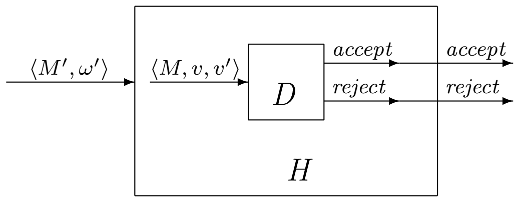


Figure 22: *Turing machine H for deciding $HALT_{TM}$*

The input to D is a Turing machine M, which is defined in Figure 23, a string $v = Ll\omega' rR$, and a string v' which can be defined in many ways, for example as v with a slightly different sequence ω'' instead of ω'. Here it is assumed that $L, l, r, R \notin \Sigma_{M'}$, that M' goes to state q_x if and only if M' halts, and that $\begin{tikzcd} q'_n \arrow[loop right] \end{tikzcd}$ $l/l{\rightarrow}$, $\forall q'_n \in Q_{M'}$.

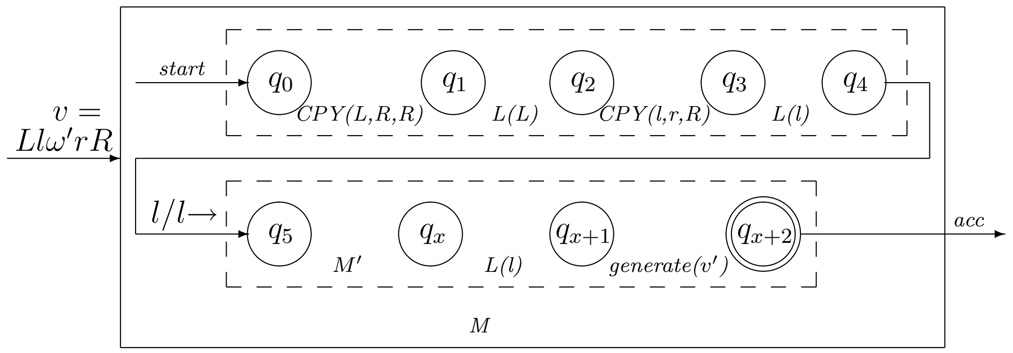


Figure 23: *Turing machine M and input string v used in Figure 22*

What M simply does is to copy v, copy ω' from inside of v, simulate the execution of M' on the copy of ω', and if M' halts generate v'. The initial self-replication forces $(M, \{v\}) \in VS$. If v' is defined appropriately, for example as suggested above, then $(M, \{v'\}) \in VS$. Thus M evolves v into v' if and only if M' halts on input ω'. In order to increase the understanding of the workings of M, the running of v on M is given in Figure 24.

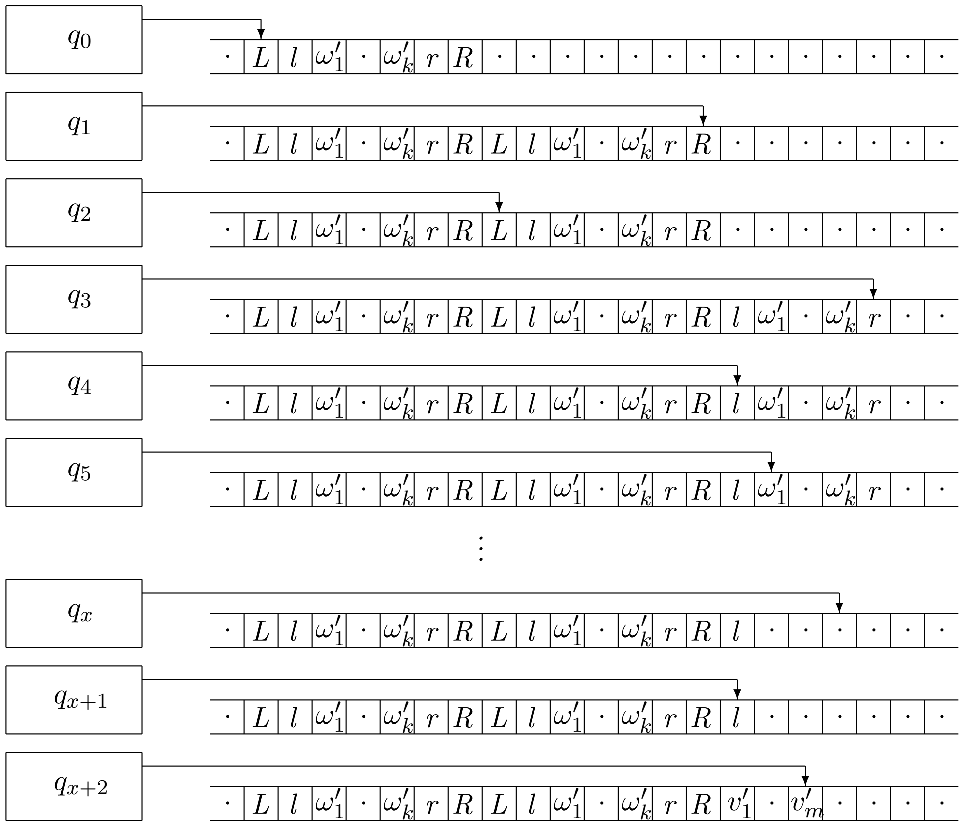


Figure 24: *Running of $Ll\omega'rR$ on M*

Q.E.D.

4.3.3 Computability

The following discussion is an interpretation of Theorem 7 in Cohen's thesis [1]. Some of the notation originates from Turing's classical paper on computable numbers [9]. Observe that the following discussion is an interpretation and might not agree with Ref. [1] and [9].

In this subsection a Turing machine M will be constructed. A class of viruses will also be specified. A virus in the given class will be such that a description of a Turing machine which computes a number is embedded in it. If $v \in V$ and $v' \in V$ are viruses in the given class, $(M, V) \in VS$, and v evolves into v' for the Turing machine M then if $\langle C_i \rangle \subset v$ then $\exists j \geq i$ such that $\langle C_j \rangle \subset v'$. Here $C_0, C_1, \ldots, C_i, \ldots C_j, \ldots$ is the configuration history for the embedded Turing machine, and will essentially contain the, so far, computed number. If $\langle C_i \rangle \subset v$ and $\langle C_{i+1} \rangle \subset v'$ then v and v' are successive members of the viral set. The successive members are called evolutions of the previous members, and thus any number that can be computed by a Turing machine can be evolved by a virus. Cohen [1] makes the following conclusion.

> We therefore conclude that "viruses" are at least as powerful a class of computing machines as TMs, and that there is a "Universal Viral Machine" which can evolve any "computable" number.

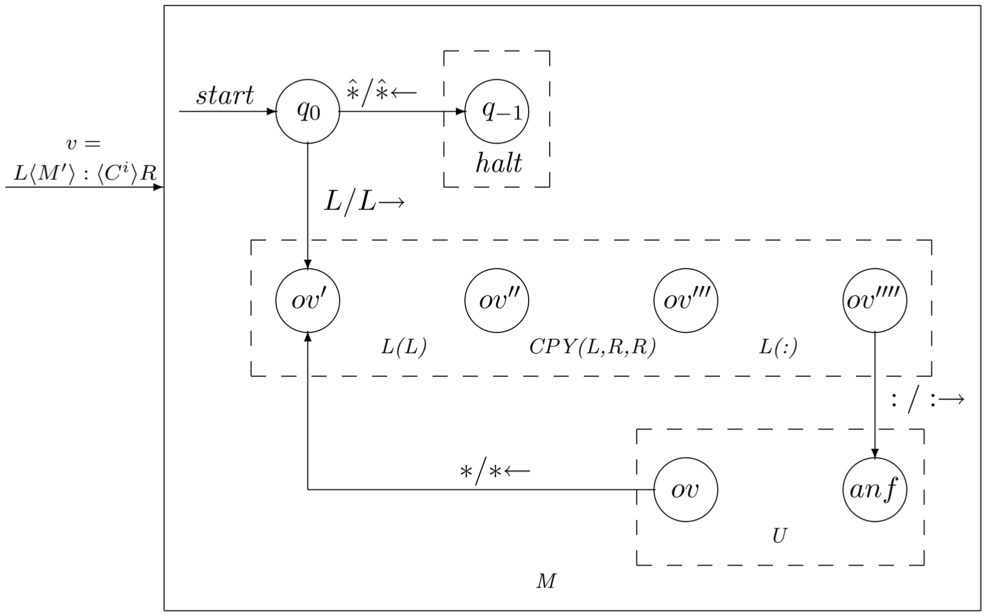
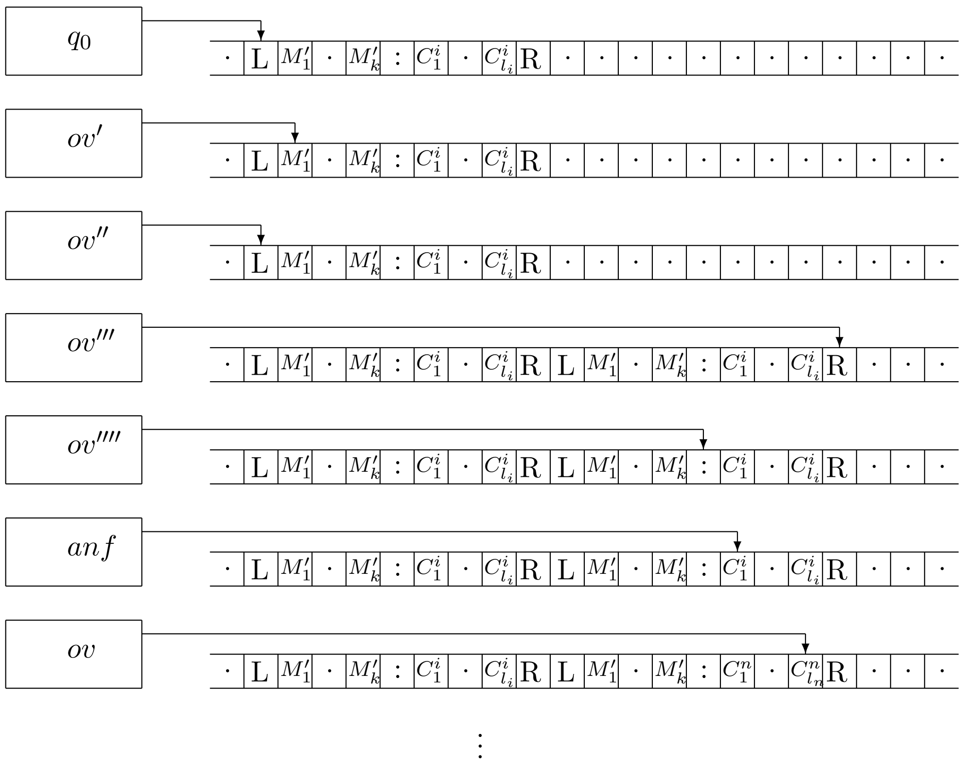


Figure 25: *Turing machine M (a possible "Universal Viral Machine")*

Now the Turing machine M in Figure 25 will be discussed. The input string is a virus starting with L and ending with R. If the input string does not start with L the machine will halt. Otherwise a copy of the virus will be written next to the original virus, to make sure that the virus really is a virus. Apart from L and R the input string contains a description of a Turing machine M' that computes a number, a colon, and a description of a configuration of M'. Observe that no input is required for M' and the starting configuration of M' is simply the start state of that machine. After the copying of the input virus has been made the tape head is moved to the colon of the copied virus. From there on the universal computing machine (denoted U in Figure 25) is operating. For a description of that, see Ref. [9]. Without getting involved in too many details it can be assumed that U by using the description of a Turing machine and by using the description of a configuration calculates the next configuration and replaces the old one with that. To ensure that the right end of the virus will not be overwritten U has to be modified so that $\forall q \in Q_U$ and for the scanned symbol equal to R, move right 1 step, write R on the tape, move left one step, and continue as before. Otherwise, L and R will not be used by U. When U is finished the process of copying etc. will start all over again, resulting in the successive members of the original input virus on the tape. In Figure 26 the running of $L\langle M'\rangle : \langle C^i\rangle R$ on M is demonstrated.



q_0
ov'
ov''
ov'''
ov''''
anf
ov
· L M'_1 · M'_k : C^i_1 · $C^i_{l_i}$ R · · · · · · · · · · · ·
· L M'_1 · M'_k : C^i_1 · $C^i_{l_i}$ R L M'_1 · M'_k : C^i_1 · $C^i_{l_i}$ R · · ·
· L M'_1 · M'_k : C^i_1 · $C^i_{l_i}$ R L M'_1 · M'_k : C^n_1 · $C^n_{l_n}$ R · · ·

Figure 26: *Running of $L\langle M'\rangle : \langle C^i\rangle R$ on M ($n = i + 1$)*

5 Summary

In this paper the necessary terminology for a formal discussion of computer viruses has been introduced. Informally a computer virus is defined as a computer program that has the ability to infect other programs by modifying them to include a, possibly evolved, copy of itself. Thus, every program that gets infected may also act as a virus and infect other programs. With the infection property, a virus can spread throughout a computer system or network.

The following pseudo-program from Cohen's thesis [1] shows how a virus might be written in a pseudo-computer language.

```
program virus :=
{1234567;

subroutine infect-executable :=
     {loop:  file = get-random-executable-file;
     if first-line-of-file = 1234567
          then goto loop;
     prepend virus to file;}

subroutine do-damage :=
     {whatever damage is to be done}

subroutine trigger-pulled :=
     {return true if some condition holds}

main-program :=
     {infect-executable;
     if trigger-pulled then do-damage;
     goto next;}

next:}
```

Figure 27: *A simple virus*

The example virus in Figure 27 searches for an uninfected executable file by looking for executable files without the "1234567" at the beginning. The virus then prepends itself to the executable turning it to an infected file. After that the virus checks if some triggering condition is true and does damage. Finally, the virus executes the rest of the program it was prepended to.

As was shown in Section 4 it is in general impossible to detect viruses. However, any particular virus can be detected by a particular detection scheme [8]. For example, the virus in Figure 27 could easily be detected by looking for "1234567" as the first line of an executable file. If found the executable would not be run and the virus would therefore not be able to spread. The program in Figure 28 is used instead of the normal run command, and refuses to execute programs infected by the virus in Figure 27.

```
program new-run-command :=
{file = name-of-program-to-be-executed;
if first-line-of-file = 1234567 then
     {print "the program has a virus";
     exit;}
otherwise run file;}
```

Figure 28: *Protection for the virus in Figure 27*

Although it is possible to protect oneself for a particular virus, any detection scheme can be circumvented. For example, if an attacker knew that a user was using the program in Figure 28 as protection from viral attack, the virus in Figure 27 could easily be substituted with a virus where the first line was changed from "1234567" to "123456". Thus it can be concluded that no infection can exist that cannot be detected, and no detection mechanism can exist that cannot be infected.

According to Cohen [8] a balance of coexistent viruses and defenses could exist, such that a given virus could only do damage to a given portion of the system, while a given protection scheme could only protect against a given set of viruses. This picture will be somewhat more involved when evolution is taken into account. In Figure 29 a modification of the virus in figure 27 to an evolutionary virus is given.

```
program evolutionary-virus :=
{ ...
subroutine print-random-statement :=
     {print random-variable-name, " = ", random-variable-name;
     loop:  if random-bit = 0 then
          {print random-operator, random-variable-name;
          goto loop;}
     print semicolon;}

subroutine copy-virus-with-random-insertions :=
     {loop:  copy evolutionary-virus to virus till semicolon-found;
     if random-bit = 1 then print-random-statement;
     if not-end-of-input-file goto loop;}

main-program :=
     {copy-virus-with-random-insertions;
     infect-executable;
     if trigger-pulled then do-damage;
     goto next;}

next:}
```

Figure 29: *An evolutionary virus*

Since all programs can evolve, the program that evolves into a difficult to attack program would, according to Cohen, more likely survive as would a virus that was more difficult to detect. A final quotation of Cohen [8]:

> As evolution takes place, balances tend to change, with the eventual result being unclear in all but the simplest circumstances. This has very strong analogies to biological theories of evolution, and might relate well to genetic theories of diseases. Similarly, the spread of viruses through systems might well be analyzed by using mathematical models used in the study of infectious diseases.

Finally, it will be concluded that a virus need not be used for destructive purposes. For example, a compression virus which finds uninfected executables, compresses them, and prepends itself to them, could be useful and, acording to Cohen [1], save over 50% of the space taken up by executable files in an average system. Upon execution, the infected program decompresses itself and executes normally.

6 References

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Apart from the references mentioned above also a numerous amount of web pages has provided information for this paper.